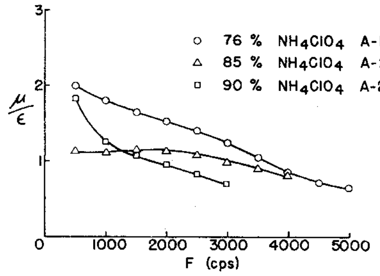

Fig. 1 Comparison of response function-frequency curves for three composite propellants differing only in oxidizer concentration.

response function. The highest oxidizer concentration examined oscillated in the end burner, but with such small amplitudes that growth and decay constants could not be measured. An estimate of the response function would be 0.3 to 0.6 between 500 and 2000 cps, whereas above 2000 cps, the value would be less.

We might speculate that the relative instability of these propellants is proportional to the rate of energy release in the total combustion zone (product of heat of explosion, linear burning rate, and density). Figure 2, which is a plot of energy release in the combustion zone ($\dot{\mathcal{E}}$) as a function of oxidizer concentration at 200 psig, shows that, based on this speculation, the propellant with 85% oxidizer should be the most unstable. Because this is not consistent with the experimental results, the speculation must be considered inadequate.

If we concentrate on more detailed aspects of the combustion, we find that the energy release in the diffusion flame zone may very well explain the results, i.e., the rate of energy release in the diffusion flame zone may be a maximum at a lower oxidizer concentration than the rate in the combustion zone as a whole, causing the 76% oxidizer system to be most unstable. This speculation implies that the diffusion flame is the responsive mechanism to pressure disturbances. This seems likely in that, of the two major energy release mechanisms, diffusion flame and oxidizer decomposition, the oxidizer decomposition has a rapid response in terms of the period of oscillation in the frequency region below about 4000 cps. Above 4000 cps, the ammonium perchlorate decomposition flame may begin to play a role in the combustion response and at higher frequencies may be the dominant response mechanism. This speculation will be examined both analytically and experimentally in the future.

Summary

The oxidizer-fuel ratio has only subtle effects on the unstable behavior of a solid composite propellant as compared to the effects of oxidizer particle size and burning-rate modifier. These subtle effects may eventually be explained in terms of the energy release in the diffusion flame zone, but to date this is only speculation.

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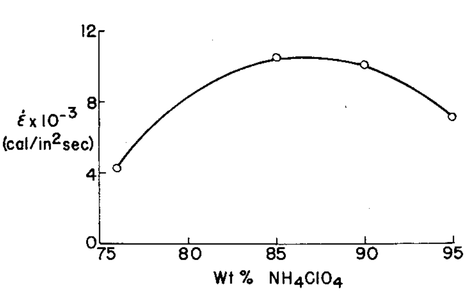

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Structure of the Boundary Layer at the Leading Edge of a Flat Plate in Hypersonic Slip Flow

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In an unpublished report[1] we have presented a solution for viscous hypersonic flow over a flat plate according to the linearized Oseen equations. The analysis was made with the thought that the linearized model might provide a valuable approximation to an otherwise intractable problem as well as serving as a guide to methods of treating the full nonlinear equations. Since the primary objective was the study of the nature of the boundary-layer/inviscid-flow interaction process all the way to the leading edge, rarefaction effects in the form of velocity slip and temperature-jump boundary conditions were an essential ingredient in the calculations. The solution showed that in the hypersonic limit the effect of these modified boundary conditions was quite radical, resulting in a flow at the leading edge which was unperturbed from the freestream state: there was "perfect slip" at the leading edge. This result means, of course, that the linearized equations should depict correctly the initial departures from freestream conditions at the leading edge. In view of the importance of this result in the wider context of the complete nonlinear formulation, it would seem to be worthwhile to present here the essence of the method used and some of the other conclusions reached. Full details are available in Ref. 1.

Method of Solution

In brief, the linearized compressible Oseen equations are represented in nondimensional form using the mean free path $\lambda \propto Re/M$ as the scaling length; here Re is the freestream Reynolds number per unit length and M the Mach number. A boundary-layer type of behavior is assumed, rates of change in the x direction along the plate being taken of order $1/M$, $M \to \infty$ compared with rates of change in the y direction normal to the plate. The Lagerstrom-Cole-Trilling method[2–4] of splitting the Oseen equations into longitudinal and transverse waves is employed, and, in the hypersonic limit $M \to \infty$, these reduce to

Pressure Wave (longitudinal)

$$\frac{\partial^2 \varphi_1}{\partial \xi^2} - \frac{\partial^2 \varphi_1}{\partial Y^2} = \beta\gamma \frac{\partial^3 \varphi_1}{\partial \xi \partial Y^2} \qquad \mathbf{q}_1 = \nabla \varphi_1 \tag{1}$$

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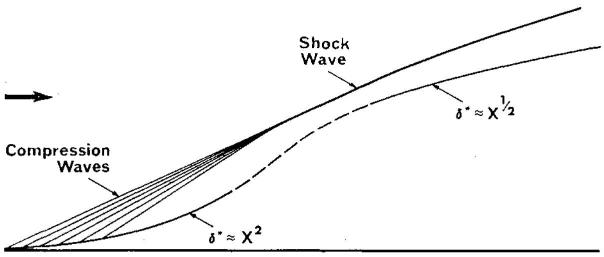

Fig. 1 Leading edge flow structure.

Thermal Wave (longitudinal)

$$\frac{\partial \varphi_2}{\partial \xi} = \beta \frac{\partial^2 \varphi_2}{\partial Y^2} \qquad \mathbf{q}_2 = \nabla \varphi_2 \tag{2}$$

Viscous Wave (transverse)

$$\frac{\partial \psi}{\partial \xi} = \frac{\partial^2 \psi}{\partial Y^2} \qquad \mathbf{q}_3 = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right) \tag{3}$$

where $\xi = xRe/M^2$, $Y = yRe/M$, with origin at the leading edge; the Prandtl number $Pr = 1/\beta$, γ is the ratio of specific heats, and $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3$ is the vector velocity.

Formal solutions of these equations, using slip boundary conditions, can be obtained readily by the method of Laplace transforms, provided it is assumed that there is no influence upstream of the leading edge. In fact, Eqs. (2) and (3) are simply the thermal and viscous boundary-layer equations for Oseen flow, and Eq. (1) is the linearized supersonic flow equation, modified by a parabolic diffusion term on the right-hand side; hence, we should indeed expect no upstream influence.

Solution near the Leading Edge

General inversion of the transformed solution is not possible, but, for the small ξ leading edge region, which is of prime concern here, we can obtain analytic representations by using asymptotics. We find that the asymptotic expansions corresponding to Eqs. (1–3) are

$$\varphi_1 = \frac{2U\gamma^{1/2}}{3Re(1-\gamma)\pi^{1/2}\beta^{1/2}}\left(\frac{1}{A_1} + \frac{\omega\beta}{C_1}\right)\xi^{3/2} \exp\left(-\frac{Y^2}{4\beta\gamma\xi}\right)\int_0^\infty \exp\left(-\frac{Y_s}{(2\beta\gamma\xi)^{1/2}} - \frac{s^2}{2}\right)s^3 ds + \ldots \tag{4}$$

$$\varphi_2 = -\frac{2U\beta^{1/2}}{\mathrm{Re}\ A_1\pi^{1/2}}\xi^{1/2}\exp\left(-\frac{Y^2}{4\beta\xi}\right) + \frac{UY}{ReA_1}\operatorname{erfc}\frac{Y}{2(\beta\xi)^{1/2}} + \ldots \tag{5}$$

$$\psi = \frac{UM}{Re\ A_1\pi^{1/2}} Y\xi^{1/2}\exp\left(-\frac{Y^2}{4\xi}\right) - \frac{UM}{Re\ A_1}\left(\xi + \frac{1}{2}Y^2\right)\operatorname{erfc}\frac{Y}{2\xi^{1/2}} + \ldots \tag{6}$$

where U is the freestream velocity, ω is the wall to freestream temperature perturbation ratio T_w/T_∞, and A_1 and C_1 are slip and temperature-jump coefficients, given by

$$A_1 = \left(\frac{\pi\gamma}{2}\right)^{1/2}\frac{2-\sigma}{\sigma} \qquad C_1 = \left(\frac{\pi\gamma}{2}\right)^{1/2}\frac{2\gamma}{\gamma - 1}\frac{2-\alpha}{\alpha}$$

in which σ and α are the momentum and thermal accomodation coefficients of the plate surface.

The three waves represented by solutions (4–6) are diffusive, so that the hyperbolic character of Eq. (1), the inviscid left-hand side of the equation, plays no part near the origin. Thus, there is no leading edge shock wave, and the region near the leading edge is one of diffusive action only. There is indeed a familiar representation of the nature of the flow which can be made here. If a displacement thickness δ^* is calculated from the thermal and viscous waves only, Eqs. (2) and (3), we find that

$$Ud\delta^*/dx = (v_1)_w \tag{7}$$

for all x, where $(v_1)_w$ is the pressure wave component of the velocity [Eq. (1)] normal to the wall. This is equivalent to the standard relationship used in small perturbation theory for flow over a body of thickness δ^*. Thus, we can regard the flow as being composed of an inner boundary layer (with constant pressure) that is given by the viscous and thermal waves, Eqs. (2) and (3), together with an external, but viscosity-dependent pressure wave, Eq. (1), representing flow about the displacement thickness of the inner layer. The concept, moreover, is valid all the way to the leading edge within the limits of the hypersonic approximation and the slip boundary condition assumption.

This fact is suggestive of an approach to obtaining a complete solution of the nonlinear equations. Thus, the nonlinear analog of Eqs. (2) and (3) are simply the compressible boundary-layer equations. However, no nonlinear equivalent to Eq. (1) has been treated, and it is, in fact, not clear what the appropriate equation is; the left-hand side of Eq. (1) can be deduced as the linearized form of hypersonic small disturbance theory, but the nonlinear term corresponding to the viscous right-hand side can only be obtained from rather dubious arguments, such as by an appeal to the unsteady flow analogy of hypersonic flow theory (see Ref. 1). Whether such an approach is possible or not, since the linearized solution equations (4–6) are valid asymptotically at the leading edge $\xi = 0$, it is clear that the latter can be used as the lead term in an expansion of the Navier-Stokes equations downstream from the leading edge. From this point of view it would seem that a major stumbling block in solving completely the viscous flow flat plate problem is removed by use of slip rather than no-slip boundary conditions, at least in the hypersonic case.

We remark here that small x means

$$\xi = Re_x/M^2 \ll 1$$

and the hypersonic approximation is valid for

$$\xi \gg 1/M^2 \qquad \text{or} \qquad Re_x \gg 1$$

Thus, the bounds of validity of the results equations (4–6) are given.

The perturbation slip velocity and surface pressure corresponding to Eqs. (4–6) are

$$\frac{u_w}{U} = -\frac{1}{A_1\pi^{1/2}}\left(\frac{Re_x}{M^2}\right)^{1/2} + O\left(\frac{Re_x}{M^2}\right) \tag{8}$$

$$\frac{P_w}{P_\infty} = \frac{2\gamma^{1/2}}{\beta^{1/2}\pi^{1/2}}\left(\frac{1}{A_1} + \frac{\omega\beta}{C_1}\right)\left(\frac{Re_x}{M^2}\right)^{1/2} + O\left(\frac{Re_x}{M^2}\right) \tag{9}$$

From Eqs. (4) and (7) we get for the displacement thickness near the origin

$$\delta^* = \frac{M}{Re(\gamma - 1)\beta}\left(\frac{1}{A_1} + \frac{\omega\beta}{C_1}\right)\left(\frac{Re_x}{M^2}\right)^2 \tag{10}$$

Equation (10) shows that, in conformity with zero upstream influence, the displacement thickness is zero at the origin; far downstream it must have the usual parabolic form. The intermediate region cannot be represented analytically; however, it is clear that the initial profile of the boundary layer leads to the formation of compression waves at the leading edge, Eq. (1), which merge some distance downstream to form a shock wave (a strong compression wave in linear theory) (see Fig. 1). Such a flow model has been suggested already from results of experimental observation.[5]

Surface Pressure

In Ref. 1 an estimate is made of the surface pressure distribution along the plate. Mention will be made here only of the rather surprising similarity of the results obtained to those of Oguchi[6] based on a nonlinear approximation and to the fact that the governing parameter is $\xi = Re_x/M^2$, in agreement with Oguchi's results and the general argument made recently by Talbot.[7]

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Integral Solution for Compressible Laminar Mixing

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AN exact solution for laminar mixing of a compressible fluid with Prandtl number one and a viscosity temperature relationship of the form $u \sim T^{\omega}$ was obtained by Chapman in Ref. 1. This note considers the solution of this problem using the approximate integral technique. The integral solution is of considerable interest since its simplicity commonly affords additional flexibility, although possibly at the expense of accuracy and uncertainty.

The differential equation of motion for similar solutions is[1]

$$-\frac{\zeta}{2}\frac{du^*}{d\zeta} = \frac{d^{[(u^*T^{*\omega-1})(du^*/d\zeta)]}}{d\zeta} \qquad (1)$$

where

$$\zeta \equiv \psi/(u_e C \nu_e x)^{1/2}$$

T^* and u^* represent the temperature and velocity nondimensionalized by the freestream values (represented by the subscript e), and ψ is the stream function. The energy equation for Prandtl number one and adiabatic conditions (although adiabatic flow has been assumed the inclusion of heat transfer would not introduce any additional complication) is

$$T^* = 1 + \frac{\gamma - 1}{2} M_e^2 - \left(\frac{\gamma - 1}{2}\right) M_e^2 u^{*2} \qquad (2)$$

The boundary conditions for a free layer with a dear-air environment are

$$\begin{aligned} \text{at } y = \infty:\ \zeta = \infty, \quad u^* = 1 \\ y = -\infty:\ u^* = 0 \end{aligned} \qquad (3)$$

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Chapman and others (e.g., Lykoudis in Ref. 2) have associated $y = -\infty$ with $\zeta = -\infty$. It is not immediately clear that $\zeta \to -\infty$, since $u^* \to 0$ as $y \to -\infty$, resulting in an indeterminant expression. In fact, as will be illustrated, the velocity distribution is such that ζ actually approaches a finite limit. This conclusion is also indicated by the velocity distributions obtained by Chapman.

In order to establish profile approximations required for an integral solution, it is desirable to first consider the series solution of Eq. (1) for large negative values of y. Assume ζ approaches a limit $-\zeta_n$, and express ζ by

$$\zeta = -\zeta_n + \Delta\zeta = -\zeta_n(1 - \epsilon) \qquad (4)$$

where ϵ is small for ζ near ζ_n. Introducing Eq. (4) into Eq. (1) gives

$$\frac{du^*}{d\epsilon} = \frac{1}{\zeta_n^2}\frac{d^{[(u^*T^{*\omega-1})(du^*/d\epsilon)]}}{d\epsilon} + \epsilon\frac{du^*}{d\epsilon} \qquad (5)$$

First, for simplicity, assume $\omega = 1.0$. Equation (5) can first be solved by neglecting the last term for $\epsilon \ll 1.0$ and using the boundary condition, $u^* = 0$ at $\epsilon = 0$. This first-order result then can be employed to approximate the last term for a higher-order solution. This approach can be continued to obtain the solution to any desired order. The result obtained after four iterations is

$$\mu_0^* = \frac{\zeta_n^2}{2}(\epsilon - 0.25\epsilon^2 + 0.01389\epsilon^3 + 0.0017\epsilon^4 + 0.0001\epsilon^5) \qquad (6)$$

The subscript 0 is used to refer to this solution of the outer region of the viscous layer. The solution converges very rapidly over the entire range from $0 \leq \epsilon \leq 1.0$. [This solution could more conveniently be obtained directly from Eq. (1) by assuming a power series solution. However, the formal solution clearly indicates the appropriate type of series required.] Clearly, Eq. (6) shows that, if ζ_n is allowed to become large, u^* will correspondingly become large (exceeding an upper limit of 1.0 demanded by the boundary condition as $y \to \infty$) even for small values of ϵ, thus precluding the possibility that $\zeta_n \to \infty$. In principle, the exact value of ζ_n must be determined by invoking the boundary condition that $u^* \to 1.0$ as $\epsilon \to \infty$. However, the series of Eq. (6) does not converge sufficiently rapidly for large values of ϵ to permit a direct evaluation of ζ_n. Chapman's results of Ref. 3 indicate that $\zeta_n = 1.233$. The values of u^* given by Eq. (6) with $\zeta_n = 1.233$ show excellent agreement with those of Ref. 3 over the range from $0 \leq u^* \leq 0.90$ (i.e., $-\zeta_n \leq \zeta \leq 1.5$).

A series solution of Eq. (1) for $\omega \neq 1.0$ can be generated in exactly the same manner by introducing T^* as given by Eq. (2). It was convenient, however, to simply introduce a power series solution similar to Eq. (6) and then evaluate the undetermined coefficients by Eq. (1). Integrating Eq. (1) twice gives [after incorporating Eq. (2)]

$$\frac{1}{2}\int_0^{\epsilon} u^* d\epsilon = -\frac{A}{2B\omega}[1 - (1 - Bu^{*2})^{\omega}] + \int_0^{\epsilon} [\textstyle\int \epsilon du^*]\, d\epsilon \qquad (7)$$

where

$$A \equiv \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\omega - 1}$$

$$B \equiv \frac{[(\gamma - 1)/2]\, M_e^2}{1 + [(\gamma - 1)/2]\, M_e^2}$$

The second term of Eq. (7) can be expanded in a Taylor's series and then the assumed power series solution introduced to evaluate the undetermined coefficients. The resulting